

FEJÉR MEANS OF VILENKIN-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to prove that there exist a martingale $f \in H_{1/2}$ such that Fejér means of Vilenkin-Fourier series of the martingale f is not uniformly bounded in the space $L_{1/2}$.

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1. INTRODUCTION

In one-dimensional case the weak type inequality

$$\mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1, \quad (\lambda > 0)$$

can be found in Zygmund [14] for the trigonometric series, in Schipp [7] for Walsh series and in Pál, Simon [6] for bounded Vilenkin series. Again in one-dimensional, Fujji [5] and Simon [9] verified that σ^* is bounded from H_1 to L_1 . Weisz [12] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the space L_p for $p > 1/2$. Simon [8] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava ([4], see also [3]).

In [3] the following is proved:

For any bounded Vilenkin system the maximal operator of the Fejér means is not bounded from the martingale Hardy space $H_{1/2}$ to the space $L_{1/2}$.

In this paper we shall prove a stronger result then the unboundedness of the maximal operator from the Hardy space $H_{1/2}$ to the space $L_{1/2}$, in particular, we shall prove that there exists a martingale $f \in H_{1/2}$ such that Fejér means of Vilenkin-Fourier series of the martingale f is not uniformly bounded in the the space $L_{1/2}$.

2. DEFINITIONS AND NOTATIONS

Let N_+ denote the set of the positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive integers, not less than 2. Denote by $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ the addition group of integers modulo m_k .

Define the group G_m as the complete direct product of the groups Z_{m_i} , with the product of the discrete topologies of Z_{m_j} 's.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k, \quad (j \in Z_{m_k})$$

is the Haar measure on G_{m_k} with $\mu(G_m) = 1$.

If $\sup_n m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. **In this paper we discuss bounded Vilenkin groups only.**

The elements of G_m represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots), \quad (x_i \in Z_{m_i}).$$

It is easy to give a base for the neighborhood of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}, \quad (x \in G_m, n \in N).$$

Denote $I_n := I_n(0)$, for $n \in N_+$.

If we define the so-called generalized number system based on m in the following way :

$$M_0 := 1, \quad M_{k+1} := m_k M_k, \quad (k \in N),$$

then every $n \in N$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$, ($j \in N_+$) and only a finite number of n_j 's differ from zero.

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. At first define the complex valued function $r_k(x) : G_m \rightarrow C$, The generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k), \quad (i^2 = -1, x \in G_m, k \in N).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in N).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 10].

Now we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\begin{aligned}
\widehat{f}(k) &: = \int_{G_m} f \overline{\psi}_k d\mu, & (k \in N), \\
S_n f &: = \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, & (n \in N_+, S_0 f := 0), \\
\sigma_n f &: = \frac{1}{n} \sum_{k=0}^{n-1} S_k f, & (n \in N_+), \\
D_n &: = \sum_{k=0}^{n-1} \psi_k, & (n \in N_+).
\end{aligned}$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, \quad (0 < p < \infty).$$

The σ -algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by F_n ($n \in N$). Denote by $f = (f^{(n)}, n \in N)$ a martingale with respect to F_n ($n \in N$). (for details see e.g. [11]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in N} |f^{(n)}|.$$

In case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n \in N} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u) d\mu(u) \right|.$$

For $0 < p < \infty$, the Hardy martingale spaces $H_p(G_m)$ consist of all martingale, for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in N)$ is a martingale.

If $f = (f^{(n)}, n \in N)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f^{(k)}(x) \overline{\Psi}_i(x) d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in N)$ obtained from f .

For a martingale f the maximal operators of the Fejér means are defined by

$$\sigma^* f(x) = \sup_{n \in N} |\sigma_n f(x)|.$$

A bounded measurable function a is p -atom, if there exists a interval I , such that

$$\begin{cases} a) & \int_I a d\mu = 0, \\ b) & \|a\|_\infty \leq \mu(I)^{\frac{1}{p}}, \\ c) & \text{supp}(a) \subset I. \end{cases}$$

3. FORMULATION OF MAIN RESULT

Theorem 1. *There exist a martingale $f \in H_{1/2}$ such that*

$$\sup_n \|\sigma_n f\|_{1/2} = +\infty.$$

Corollary 1. *There exist a martingale $f \in H_{1/2}$ such that*

$$\|\sigma^* f\|_{1/2} = +\infty.$$

4. AUXILIARY PROPOSITIONS

Lemma 1. [13] *A martingale $f = (f^{(n)}, n \in N)$ is in H_p ($0 < p \leq 1$) if and only if there exist a sequence $(a_k, k \in N)$ of p -atoms and a sequence $(\mu_k, k \in N)$, of a real numbers, such that for every $n \in N$:*

$$(1) \quad \begin{aligned} \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k &= f^{(n)}, \\ \sum_{k=0}^{\infty} |\mu_k|^p &< \infty. \end{aligned}$$

Moreover, $\|f\|_{H_p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p}$, where the infimum is taken over all decomposition of f of the form (1).

Lemma 2. [2] *Let $2 < A \in N_+$, $k \leq s < A$ and $q_A = M_{2A} + M_{2A-2} + \dots + M_2 + M_0$, then*

$$q_{A-1} |K_{q_{A-1}}(x)| \geq \frac{M_{2k} M_{2s}}{4}.$$

for

$$x \in I_{2A} (0, \dots, x_{2k} \neq 0, 0, \dots, 0, x_{2s} \neq 0, x_{2s+1}, \dots, x_{2A-1}),$$

$$k = 0, 1, \dots, A-3, \quad s = k+2, k+3, \dots, A-1.$$

5. PROOF OF THE THEOREM

Let $\{\alpha_k : k \in N\}$ be an increasing sequence of the positive integers such that:

$$(2) \quad \sum_{k=0}^{\infty} \alpha_k^{-1/2} < \infty,$$

$$(3) \quad \sum_{\eta=0}^{k-1} \frac{(M_{2\alpha_\eta})^2}{\alpha_\eta} < \frac{(M_{2\alpha_k})^2}{\alpha_k},$$

$$(4) \quad \frac{32M (M_{2\alpha_{k-1}})^2}{\alpha_{k-1}} < \frac{M_{\alpha_k}}{\alpha_k},$$

where $M = \sup \{m_0, m_1 \dots\}$, $(2 \leq M < \infty)$.

We note that such an increasing sequence $\{\alpha_k : k \in N\}$ which satisfies conditions (2-4) can be constructed.

Let

$$f^{(A)}(x) = \sum_{\{k; 2\alpha_k < A\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{1}{\alpha_k},$$

and

$$a_k(x) = \frac{M_{2\alpha_k}}{M} \left(D_{M(2\alpha_k+1)}(x) - D_{M_{2\alpha_k}}(x) \right).$$

It is easy to show that the martingale $f = (f^{(1)}, f^{(2)} \dots f^{(A)} \dots) \in H_{1/2}$.

Indeed, since

$$(5) \quad S_{M_A} a_k(x) = \begin{cases} a_k(x), & 2\alpha_k < A, \\ 0, & 2\alpha_k \geq A, \end{cases}$$

$$\begin{aligned} \text{supp}(a_k) &= I_{2\alpha_k}, \\ \int_{I_{2\alpha_k}} a_k d\mu &= 0 \end{aligned}$$

and

$$\|a_k\|_\infty \leq \frac{M_{2\alpha_k}}{M} M_{2\alpha_k+1} \leq (M_{2\alpha_k})^2 = (\text{supp } a_k)^{-2}.$$

if we apply lemma 1 and (2) we conclude that $f \in H_{1/2}$.

It is easy to show that

$$(6) \quad \widehat{f}(j) = \begin{cases} \frac{1}{M} \frac{M_{2\alpha_k}}{\alpha_k}, & \text{if } j \in \{M_{2\alpha_k}, \dots, M_{2\alpha_k+1} - 1\}, \quad k = 0, 1, 2, \dots, \\ 0, & \text{if } j \notin \bigcup_{k=1}^{\infty} \{M_{2\alpha_k}, \dots, M_{2\alpha_k+1} - 1\}. \end{cases}$$

We can write

$$(7) \quad \sigma_{q_{\alpha_k}} f(x) = \frac{1}{q_{\alpha_k}} \sum_{j=0}^{M_{2\alpha_k}-1} S_j f(x) + \frac{1}{q_{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}-1} S_j f(x) = I + II.$$

Let $M_{2\alpha_k} \leq j < q_{\alpha_k}$. Then applying (6) we have

$$(8) \quad \begin{aligned} S_j f(x) &= \sum_{v=0}^{M_{2\alpha_{k-1}+1}-1} \widehat{f}(v) \psi_v(x) + \sum_{v=M_{2\alpha_k}}^{j-1} \widehat{f}(v) \psi_v(x) \\ &= \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_{\eta+1}}-1} \widehat{f}(v) \psi_v(x) + \sum_{v=M_{2\alpha_k}}^{j-1} \widehat{f}(v) \psi_v(x) \\ &= \frac{1}{M} \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_{\eta+1}}-1} \frac{M_{2\alpha_\eta}}{\alpha_\eta} \psi_v(x) + \frac{1}{M} \frac{M_{2\alpha_k}}{\alpha_k} \sum_{v=M_{2\alpha_k}}^{j-1} \psi_v(x) \\ &= \frac{1}{M} \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}}{\alpha_\eta} \left(D_{M_{2\alpha_{\eta+1}}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \\ &\quad + \frac{1}{M} \frac{M_{2\alpha_k}}{\alpha_k} \left(D_j(x) - D_{M_{2\alpha_k}}(x) \right). \end{aligned}$$

Applying (8) in II we have

$$\begin{aligned} II &= \frac{1}{M} \frac{q_{\alpha_k} - M_{2\alpha_k}}{q_{\alpha_k}} \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}}{\alpha_\eta} \left(D_{M_{2\alpha_{\eta+1}}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \\ &\quad + \frac{1}{M} \frac{M_{2\alpha_k}}{\alpha_k q_{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}-1} \left(D_j(x) - D_{M_{2\alpha_k}}(x) \right) \\ &= II_1 + II_2. \end{aligned}$$

It is evident

$$\left| \frac{q_{\alpha_k} - M_{2\alpha_k}}{q_{\alpha_k}} \right| < 1$$

and

$$\begin{aligned} & \left| \left(D_{M_{2\alpha_\eta+1}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \right| \\ & \leq M_{2\alpha_\eta+1} = m_{2\alpha_\eta} M_{2\alpha_\eta} \leq M \cdot M_{2\alpha_\eta}. \end{aligned}$$

Applying (3) we have

$$(9) \quad |II_1| \leq \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}}{\alpha_\eta} \frac{1}{M} M \cdot M_{2\alpha_\eta} \leq \frac{2(M_{2\alpha_{k-1}})^2}{\alpha_{k-1}}.$$

Since

$$D_{j+M_{2\alpha_k}}(x) = D_{M_{2\alpha_k}}(x) + \psi_{M_{2\alpha_k}}(x) D_j(x), \quad \text{when } j < M_{2\alpha_k}.$$

for II_2 we have:

$$\begin{aligned} |II_2| &= \frac{1}{M} \frac{M_{2\alpha_k}}{\alpha_k \cdot q_{\alpha_k}} \left| \sum_{j=0}^{q_{\alpha_k}-1} D_{j+M_{2\alpha_k}}(x) - D_{M_{2\alpha_k}}(x) \right| \\ &= \frac{1}{M} \frac{M_{2\alpha_k}}{\alpha_k \cdot q_{\alpha_k}} \left| \psi_{M_{2\alpha_k}}(x) \sum_{j=0}^{q_{\alpha_k}-1} D_j(x) \right| \\ &= \frac{1}{M} \frac{M_{2\alpha_k}}{q_{\alpha_k}} \frac{q_{\alpha_k}-1}{\alpha_k} |K_{q_{\alpha_k}-1}(x)| \\ &\geq \frac{1}{2M} \frac{q_{\alpha_k}-1}{\alpha_k} |K_{q_{\alpha_k}-1}(x)|. \end{aligned}$$

Since

$$q_{\alpha_k} \leq M_{2\alpha_k} \left(1 + \frac{1}{4} + \dots + \frac{1}{4^n} \right) \leq 2M_{2\alpha_k},$$

for II_2 we obtain

$$|II_2| \geq \frac{1}{2M} \frac{q_{\alpha_k}-1}{\alpha_k} |K_{q_{\alpha_k}-1}(x)|.$$

Let $M_{2\alpha_{k-1}+1} - 1 \leq j < M_{2\alpha_k}$. Then from (8) we have

$$\begin{aligned}
|S_j f(x)| &= \left| \sum_{v=0}^{j-1} \widehat{f}(v) \psi_v(x) \right| \\
&= \left| \sum_{v=0}^{M_{2\alpha_{k-1}+1}-1} \widehat{f}(v) \psi_v(x) \right| \\
&= \left| \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_{\eta+1}}-1} \frac{M_{2\alpha_\eta}}{M \cdot \alpha_\eta} \psi_v(x) \right| \\
&= \left| \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}}{M \cdot \alpha_\eta} \left(D_{M_{2\alpha_{\eta+1}}}(x) - D_{M_{2\alpha_\eta}}(x) \right) \right| \\
&\leq \frac{2(M_{2\alpha_{k-1}})^2}{\alpha_{k-1}}.
\end{aligned}$$

Hence

$$\begin{aligned}
(10) \quad |I| &\leq \frac{1}{q_{\alpha_k}} \sum_{j=0}^{M_{2\alpha_k}-1} |S_j f(x)| \\
&\leq \frac{2M_{2\alpha_k}}{q_{\alpha_k}} \frac{(M_{2\alpha_{k-1}})^2}{\alpha_{k-1}} \\
&\leq \frac{2(M_{2\alpha_{k-1}})^2}{\alpha_{k-1}}.
\end{aligned}$$

Applying (4) we have

$$|I|, |II_1| \leq \frac{2(M_{2\alpha_{k-1}})^2}{\alpha_{k-1}} \leq \frac{1}{16M} \frac{M_{\alpha_k}}{\alpha_k}.$$

Consequently,

$$\begin{aligned}
(11) \quad |\sigma_{q_{\alpha_k}} f(x)| &\geq |II_2| - (|I| + |II_1|) \\
&\geq \frac{1}{8M \cdot \alpha_k} \left(4q_{\alpha_{k-1}} \left| K_{q_{\alpha_{k-1}}}(x) \right| - M_{\alpha_k} \right).
\end{aligned}$$

Denote

$$I_{2\alpha_k}(0, \dots, x_{2\eta} \neq 0, 0, \dots, 0, x_{2s} \neq 0, x_{2s+1}, \dots, x_{2\alpha_k-1}) = I_{2\alpha_k}^{\eta,s}.$$

Let

$$x \in I_{2\alpha_k}^{\eta,s}, \quad \eta = \left\lfloor \frac{\alpha_k}{2} \right\rfloor, \left\lfloor \frac{\alpha_k}{2} \right\rfloor + 1, \dots, \alpha_k - 3, \quad s = \eta + 2, \eta + 3, \alpha_k - 1.$$

Applying lemma 2 we have:

$$4q_{\alpha_k-1} \left| K_{q_{\alpha_k-1}}(x) \right| \geq M_{2\eta} M_{2s}.$$

Since

$$2s \geq 2 \left\lceil \frac{\alpha_k}{2} \right\rceil + 4 > \alpha_k + 1,$$

we have

$$M_{2s} > M_{\alpha_k+1} \geq m_{\alpha_k} M_{\alpha_k} \geq 2M_{\alpha_k}.$$

Hence

$$(12) \quad M_{2s} M_{2\eta} - M_{\alpha_k} \geq \frac{1}{M} M_{2s} M_{2\eta}.$$

From (11-12) we have

$$\left| \sigma_{q_{\alpha_k}} f(x) \right| \geq \frac{1}{8M^2 \cdot \alpha_k} M_{2s} M_{2\eta}, \quad x \in I_{2\alpha_k}^{\eta,s},$$

where

$$\eta = \left\lceil \frac{\alpha_k}{2} \right\rceil, \left\lceil \frac{\alpha_k}{2} \right\rceil + 1, \dots, \alpha_k - 3, \quad s = \eta + 2, \eta + 3, \alpha_k - 1.$$

Hence we can write

$$\begin{aligned} & \int_{G_m} \left| \sigma_{q_{\alpha_k}} f(x) \right|^{\frac{1}{2}} d\mu(x) \\ & \geq \sum_{\eta=\lceil \alpha_k/2 \rceil}^{\alpha_k-3} \sum_{s=\eta+2}^{\alpha_k-1} \sum_{x_{2s+1}=0}^{m_{2s+1}-1} \dots \sum_{x_{2\alpha_k-1}=0}^{m_{2\alpha_k-1}-1} \int_{I_{2\alpha_k}^{\eta,s}} \left| \sigma_{q_{\alpha_k}} f(x) \right|^{\frac{1}{2}} d\mu(x) \\ & \geq \frac{1}{\sqrt{8}M\sqrt{\alpha_k}} \sum_{\eta=\lceil \alpha_k/2 \rceil}^{\alpha_k-3} \sum_{s=\eta+2}^{\alpha_k-1} \frac{m_{2s+1} \dots m_{2\alpha_k-1}}{M_{2\alpha_k}} \sqrt{M_{2s} M_{2\eta}} \\ & \geq \frac{1}{8M\sqrt{\alpha_k}} \sum_{\eta=\lceil \alpha_k/2 \rceil}^{\alpha_k-3} \sum_{s=\eta+2}^{\alpha_k-1} \frac{\sqrt{M_{2s} M_{2\eta}}}{M_{2s+1}} \\ & \geq \frac{1}{8M^2\sqrt{\alpha_k}} \sum_{\eta=\lceil \alpha_k/2 \rceil}^{\alpha_k-3} \sum_{s=\eta+2}^{\alpha_k-1} \sqrt{\frac{M_{2\eta}}{M_{2s}}}. \end{aligned}$$

It is easy to show that

$$\sum_{s=\eta+2}^{\alpha_k-1} \sqrt{\frac{M_{2\eta}}{M_{2s}}} \geq \sqrt{\frac{M_{2\eta}}{M_{2\eta+4}}} \geq \frac{1}{M^2}.$$

Consequently,

$$\begin{aligned}
& \int_G |\sigma_{q_{\alpha_k}} f(x)|^{\frac{1}{2}} d\mu(x) \\
& \geq \frac{1}{8M^2 \sqrt{\alpha_k}} \sum_{\eta=[\alpha_k/2]}^{\alpha_k-3} \left(\sum_{s=\eta+2}^{\alpha_k-1} \sqrt{\frac{M_{2\eta}}{M_{2s}}} \right) \\
& \geq \frac{1}{8M^4 \sqrt{\alpha_k}} \sum_{\eta=[\alpha_k/2]}^{\alpha_k-3} 1 \\
& \geq c\sqrt{\alpha_k} \rightarrow \infty, \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

Theorem 1 is proved.

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